B.) 'Least Action' and the Energy Principle in MHD

→ Introduction

- we now arrive at the $\boxed{\text{MHD Energy Principle}}$, which is a highlight of MHD, plasma physics and classical physics, in general.

- Energy Principle → stability

i.e. till now
- 218B – waves, etc.
- 218A – trivial instabilities (i.e. 2-stream, bump-on-tail, J-driven ion-acoustic)

realistic plasmas {lab or astro} → free energy (∇P, ∇J etc.) + complex geometry, b.c.'s, etc.

→ instabilities with complex dynamics ...

i.e.
- Rayleigh-Benard → ∇S
- Interchanges → K, ∇P (includes Rayleigh-Taylor)
- kinks, tearing → ∇J, $q(r)$

→ { relaxation, turbulence, shocks ...
limits on performance (lab)
restrictions on morphology (lab and astro) }

- brute force, frontal assault on instabilities often leads to heavy casualties ....

$\therefore$

- need a simple criterion, i.e. a necessary/sufficient criterion to identify and characterize instabilities

$\Rightarrow$ Energy Principle !

- Energy Principle is very much in spirit of R-R Variational principle $\Leftrightarrow$ no surprise as both based on self-adjointness of linear operator

write $\underline{v}_0 = 0$ force eq

- Proceed via: $\rho \frac{\partial v}{\partial t} = \rho_0 \frac{\partial^2 \xi}{\partial t^2} =$

- sketch of Principle of Least Action for Ideal MHD $\Rightarrow$ Lagrangian formulation (Kulsrud 4.7)

N.B. This underlies formulation in terms of displacement...

- MHD eigenmode equation (generalizes simple wave studies so far), second order W

$\Rightarrow$

- energy principle. (Kulsrud 7.1, 7.2) (Kadomtsev Article)

— applications (various)

i.) Principle of Least Action for MHD

— For ideal MHD, can immediately write

$$L = \int d^3x \left[\frac{\rho v^2}{2} \right] - W \qquad \text{(Lagrangian)}$$

$$W = \int d^3x \left(\frac{P}{\gamma - 1} + \frac{B^2}{8\pi} + \rho\phi \right)$$

$S = \int dt\, L$
↳ action

$$\sim \quad \mathcal{L} = \frac{\rho v^2}{2} - \left(\frac{P}{\gamma - 1} + \frac{B^2}{8\pi} + \rho\phi \right)$$

and can derive MHD equations by $\delta L = 0$
i.e. Principle of Least Action

— key point: how parametrize trajectory variations??

i.e. for string: (easy)

$$S = \int dt \int_0^{L_0} dx \left[\frac{1}{2}\left(\frac{\partial y}{\partial t}\right)^2 - T\left[\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{1/2} - 1\right]\right]$$

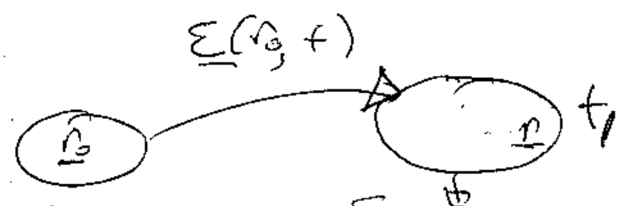
$$\delta L = \delta L/\delta y \Rightarrow \frac{\partial L}{\partial y_t}\delta y_t + \frac{\partial L}{\partial y_x}\delta y_x \quad \text{etc...}$$

∴ analogy with string suggests displacement is!

→ natural way to formulate Least Action for ideal MHD

→ natural link of MHD dynamics to particle dynamics

i.e.



[at t_1, original blob has
$\underline{r} = \underline{r}_0 + \underline{\xi}(\underline{r}_0, t)$]

— how relate $\underline{\xi}(\underline{r}_0, t)$ to Eulerian velocity?

i.e. during δt, fluid element moves

from $\underline{r} = \underline{r}_0 + \underline{\xi}(\underline{r}_0, t)$ → to $\underline{r}_0 + \underline{\xi}(\underline{r}_0, t) + \left(\partial\underline{\xi}/\partial t\right)\delta t$

$$\therefore \quad \underline{V}(\underline{r}_0 + \underline{\xi}(\underline{r}_0, t), t) = \frac{\partial \underline{\xi}}{\partial t}(\underline{r}_0, t)$$

→ 3 components of $\underline{\xi}$ satisfy 3 nonlinear odes with $\underline{\xi}(\underline{r}_0, t_0) = 0$ as i.c.

$\rightarrow$ theory of ode's assures solution exists.

Now, as in wave theory, can write all changes in MHD quantities in terms of displacements, i.e.

$$\delta\rho = -\underline{\nabla}\cdot\left[\rho(\underline{r},t)\,\delta\underline{\xi}(\underline{r},t)\right]$$

$$\delta P = -\gamma P(\underline{r},t)\,\underline{\nabla}\cdot\delta\underline{\xi}(\underline{r},t) - \delta\underline{\xi}(\underline{r},t)\cdot\underline{\nabla}P(\underline{r},t)$$

$$\delta\underline{B} = \underline{\nabla}\times\left(\delta\underline{\xi}(\underline{r},t)\times\underline{B}(\underline{r},t)\right)$$

and

$$\delta V(\underline{r},t) = \underline{V}(\underline{r},t)\cdot\nabla\,\delta\underline{\xi}(\underline{r},t) - \delta\underline{\xi}(\underline{r},t)\cdot\underline{\nabla}\,\underline{V}(\underline{r},t) + \partial\,\delta\underline{\xi}(\underline{r},t)/\partial t$$

so now, can consider δS

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x\,\delta\mathcal{L}$$

$$= \int_{t_1}^{t_2} dt \int d^3x \left(\delta\rho\,\frac{\underline{V}^2}{2} + \rho\,\underline{V}\cdot\delta\underline{V} - \frac{\delta P}{\gamma-1} - \frac{\underline{B}\cdot\delta\underline{B}}{4\pi} - \delta\rho\,\phi\right)$$

plugging in δ quantities $\Rightarrow$

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x \left\{ \overbrace{\nabla\cdot(-\rho\,\delta\underline{\xi})\frac{V^2}{2} + \rho\underline{V}\cdot\left(\underline{V}\cdot\underline{\nabla}\delta\underline{\xi} - \delta\underline{\xi}\cdot\underline{\nabla}\underline{V} + \frac{\partial\delta\underline{\xi}}{\partial t}\right)}^{\delta KE} \right\} + \overbrace{\int_{t_1}^{t_2} dt\, d^3x \left(\frac{\gamma p\, \underline{\nabla}\cdot\delta\underline{\xi} + \delta\underline{\xi}\cdot\underline{\nabla}p}{\gamma-1} \right)}^{\delta ThE}$$

$$- \overbrace{\int_{t_1}^{t_2} dt \int d^3x\, \frac{\underline{B}\cdot\underline{\nabla}\times(\delta\underline{\xi}\times\underline{B})}{4\pi}}^{\delta E_B} + \int_{t_1}^{t_2} dt \int d^3x\, \nabla\cdot(\rho\,\delta\underline{\xi})\,\phi$$

Now $\delta\underline{\xi}\Big|_{t_1,t_2} = 0$, $\delta\underline{\xi}\Big|_{\text{bndry}} = 0$

bndry $\rightarrow$ compt restriction

so ibp a lot $\Rightarrow$ (with b.c.'s)

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x \left\{ \delta\underline{\xi}\cdot\left[\underset{\nabla P}{\rho\nabla\frac{\underline{V}^2}{2}} - \underline{\nabla}\cdot(\rho\underline{V}\,\underline{V}) - \underset{\rho\nabla\phi}{\rho\underline{\nabla}\frac{V^2}{2}} - \frac{\partial}{\partial t}(\rho\underline{V}) \right] - \frac{\delta\underline{\xi}\cdot\gamma\underline{\nabla}p + \delta\underline{\xi}\cdot\nabla p}{(\gamma-1)} - \delta\underline{\xi}\cdot\rho\underline{\nabla}\phi \right.$$

$$\left. + \delta\underline{\xi}\cdot\underset{\underline{J}\times\underline{B}}{\frac{(\nabla\times\underline{B})\times\underline{B}}{4\pi}} \right\}$$

so

$$\delta S = -\int_{t_1}^{t_2} dt \int d^3x \; \delta \underline{\xi} \cdot \left[\frac{\partial}{\partial t}(\rho \underline{V}) + \nabla \cdot (\rho \underline{V}\,\underline{V}) + \underline{\nabla} p - \underline{J} \times \underline{B} + \rho \underline{\nabla} \phi \right]$$

so

$$\delta S = 0 \quad \text{and} \quad \delta \underline{\xi} \neq 0 \Rightarrow$$

$$\frac{\partial}{\partial t}(\rho \underline{V}) + \underline{\nabla} \cdot (\rho \underline{V}\,\underline{V}) = -\underline{\nabla} p + \underline{J} \times \underline{B} - \rho \underline{\nabla} \phi$$

and $\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\rho \underline{V}) \Rightarrow$

$$\boxed{\rho \left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{V} \right] = -\underline{\nabla} p + \underline{J} \times \underline{B} - \rho \underline{\nabla} \phi}$$

$\Rightarrow$ equation of motion of ideal MHD emerges as "Lagrange's Equation".

Note: for case of $\underline{V} = 0 \rightarrow$ equilibrium solution then $\delta S = 0$ gives:

$$\underline{\nabla} p = \underline{J} \times \underline{B} - \rho \underline{\nabla} \phi$$

Moral of this story:

→ can derive MHD equations from Principle of Least Action

→ displacement is a useful way to formulate ideal MHD dynamics

Now this brings us to:

ii.) Energy Principle – Simple Form

Consider inhomogeneous static equilibrium/initial state with:

$$\text{eqbm conditions} \begin{cases} \underline{\nabla} P_0 = \underline{J}_0 \times \underline{B}_0 - \rho_0 \underline{g} \quad \leftarrow \text{const } g, \ (c \to 1) \\ \underline{\nabla} \times \underline{B}_0 = \frac{4\pi}{c} \underline{J}_0 \\ \underline{\nabla} \cdot \underline{B}_0 = 0 \end{cases}$$

N.b. gravity will be dropped in places

all $\begin{cases} P_0 = P_0(r_0) \\ \text{time independent} \\ \text{but inhomogeneous!} \end{cases}$

and no flow or $\left\{ \begin{array}{l} \text{self-} \\ \text{gravity} \end{array} \right.$ ....

Further assume → rigid wall bounds system (!!)

$$\to \underline{V} \cdot \hat{n} \Big|_{wall} = 0$$

$$\underline{B} \cdot \hat{n} \Big|_{wall} = 0$$

and now ... $\Rightarrow$ perturb system from eqbrm by $\underline{\xi}$

$\Rightarrow$ so, at $t=0$:

$$\underline{\xi}(r) = \underline{\xi}_0(\underline{r})$$

$$\frac{\partial \underline{\xi}(r)}{\partial t} = \frac{\partial \hat{\underline{\xi}}_0}{\partial t}(r)$$

$\Rightarrow$ keep only linear terms in $\underline{\xi}$ $\Rightarrow$

$$\underline{r} = \underline{r}_0 + \xi(\underline{r}_0, t)$$

and $\underline{r}_0 \to \underline{r}$ in argument of perturbed quantities.

<u>So</u>

$$\Rightarrow \quad \rho(t, r) = \rho_0 - \nabla\cdot(\rho_0 \underline{\xi})$$

$$P(t, \underline{r}) = P_0 - \gamma P_0 \underline{\nabla}\cdot\underline{\xi} - \underline{\xi}\cdot\underline{\nabla} P_0$$

$$\underline{B}(t, \underline{r}) = \underline{B}_0 + \underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)$$

$$4\pi J(\underline{r}, t) = \underline{j}_0 + \underline{\nabla}\times\left[\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)\right]$$

$\therefore$ putting it into equation of motion (linearized) $\Rightarrow$

$$\boxed{\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = \underline{F}(\underline{\xi})}$$

where:

$$\delta \underline{J} \times \underline{B}_0$$

$$F(\xi) = \frac{1}{4\pi}\left[\underline{\nabla}\times\left[\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)\right]\right]\times B_0$$

$$\underline{J}\times\delta\underline{B} \qquad\qquad -g\,\delta\rho$$

$$+\ \underline{J}_0 \times\left[\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)\right] \quad - \underline{g}\ \underline{\nabla}\cdot(\rho_0\underline{\xi})$$

$$+\ \underline{\nabla}\left[\underline{\xi}\cdot\underline{\nabla}P_0 + \gamma P_0(\underline{\nabla}\cdot\underline{\xi})\right]$$

$$-\underline{\nabla}P$$

with b.c. $\begin{cases} \underline{\xi}\cdot\vec{n} = 0 & \text{on surface} \\ \underline{B}\cdot\vec{n} = 0 & \text{on surface} \end{cases}$

Key Point:

$\rightarrow$ $\boxed{\underline{F}(\underline{\xi}) \text{ is self-adjoint}}$!!

i.e.

$$\int d^3x\ \underline{\eta}\cdot\underline{F}(\xi) = \int d^3x\ \underline{\xi}\cdot F(\underline{\eta})$$

→ to prove: see Kulsrud, Pblm. 6 (coming on Pblm Set III)

or consider the following (an indirect proof) ....
legerdemain involved...

→ can write total energy, to second order (in displacement) as:

i.e. $E = \left(\int d^3x \, \frac{\rho_0(\underline{r})}{2} \left(\frac{\partial \underline{\xi}}{\partial t} \right)^2 \right) + W(\underline{\xi}, \underline{\xi})$

2nd order bit of:

$$\int \left(\frac{P}{\gamma - 1} + \frac{B^2}{8\pi} + \rho\phi \right) d^3x$$

$W = W_0 + W_1 + W_2$

Key → use of $\frac{dE}{dt} = 0$, for ideal MHD (where E-cons. crucial)

Now:

→ $W = W_0 + W_1(\underline{\xi}) + W_2(\underline{\xi}, \underline{\xi})$

first order — second order

→ total energy is conserved, for any $\underline{\xi}$ with initial conditions $\underline{\xi}_0$, $\dot{\underline{\xi}}_0$,

provided $\underline{\xi} \cdot \hat{n} = \dot{\underline{\xi}} \cdot \hat{n} = 0$ (b.c.)

$dE/dt = 0$, all time

Now, $dE/dt = 0 \Rightarrow$

$$\frac{dE}{dt} = \int_V d^3x\, \rho_0 \left\{ \frac{\partial \underline{\xi}}{\partial t} \cdot \frac{\partial^2 \underline{\xi}}{\partial t^2} \right\} + W_1\left(\frac{\partial \underline{\xi}}{\partial t}\right)$$

$$+ W_2\left(\frac{\partial \underline{\xi}}{\partial t}, \underline{\xi}\right) + W_2\left(\underline{\xi}, \frac{\partial \underline{\xi}}{\partial t}\right) = 0$$

and $\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = \underline{F}(\xi) \Rightarrow$

$$\frac{dE}{dt} = \int d^3x \left[\frac{\partial \underline{\xi}}{\partial t} \cdot \underline{F}(\underline{\xi}) \right] + W_1\left(\frac{\partial \xi}{\partial t}\right)$$

$$+ W_2\left(\frac{\partial \underline{\xi}}{\partial t}, \underline{\xi}\right) + W_2\left(\underline{\xi}, \frac{\partial \underline{\xi}}{\partial t}\right)$$

but since $dE/dt = 0$ is always true, it is true at $t=0$, a particular time

setting $\dot{\underline{\xi}}_0 \equiv \underline{\eta} \Rightarrow$

$\hookrightarrow$ a particular displ. ...

$$\int d^3x\, \underline{\eta} \cdot \underline{F}(\underline{\xi}) + W_1(\eta) + W_2(\underline{\eta}, \underline{\xi})$$

$$+ W_2(\underline{\xi}, \underline{\eta}) = 0$$

now, $W_1(\underline{\eta}) = 0$ so (no velocity dependence on i.c. critical dissal only)

$$\int d^3x \left(\underline{\eta} \cdot F(\underline{\xi})\right) + \left[W_2(\underline{\eta}, \underline{\xi}) + W_2(\underline{\xi}, \underline{\eta})\right] = 0$$

or more clearly ⇒

$$\int d^3x \; \underline{\eta} \cdot F(\underline{\xi}) = -\left[W_2(\underline{\eta}, \underline{\xi}) + W_2(\underline{\xi}, \underline{\eta})\right]$$

so RHS symmetric under $\underline{\eta} \leftrightarrow \underline{\xi}$ interchange

so so is LHS ! i.e.

$$\int d^3x \; \underline{\eta} \cdot F(\underline{\xi}) = \int d^3x \; \underline{\xi} \cdot F(\underline{\eta})$$

and have proved self-adjointness !

→ finally, useful to note that if now $\underline{\eta} = \underline{\xi}$

$$\boxed{W_2(\underline{\xi}, \underline{\xi}) = -\frac{1}{2}\int d^3x \left[\underline{\xi} \cdot F(\underline{\xi})\right]}$$

— a handy expression for W_2 in terms of F !

so now, have shown that:

→ $\underline{F}(\underline{\xi})$ self-adjoint

→ $W_2(\underline{\xi})$, the potential energy of displacement $\underline{\xi}$, can be expressed as:

$$W_2(\underline{\xi}) = -\frac{1}{2}\int d^3x \left[\underline{\xi}\cdot \underline{F}(\underline{\xi})\right]$$

From these, we show several important results:

· reality of ω^2 and "exchange of stabilities"
↔ clue to structure of instability in ideal MHD

- orthogonality of eigenfunctions

- Variational structure

i) reality of ω^2, "exchange of stabilities"

$$\underline{\xi} = \tilde{\xi}(x)\, e^{-i\omega t}$$

$$-\rho_0\omega^2\,\underline{\xi} = \underline{F}(\underline{\xi}) \qquad (1)$$

$$-\rho_0\omega^{2*}\,\underline{\xi}^* = \underline{F}(\underline{\xi}^*) \qquad (2)$$

n.b. F is explicitly real

$\underline{\xi}^* * (1) - \underline{\xi} * (2) \Rightarrow$

$$-\rho_0(\omega^2-\omega^{2*})\,\underline{\xi}^*\cdot\underline{\xi} = \underline{\xi}^*\cdot\underline{F}(\underline{\xi}) - \underline{\xi}\cdot\underline{F}(\underline{\xi}^*)$$

and integrating $\Rightarrow$

$$-\rho_0(\omega^2-\omega^{2*})\int d^3x\,(\underline{\xi}^*\cdot\underline{\xi}) = \int d^3x\left[\underline{\xi}^*\cdot\underline{F}(\underline{\xi}) - \underline{\xi}\cdot\underline{F}(\underline{\xi}^*)\right]$$

$= 0$, by self-adjoint property

so $\underline{\xi}^*\cdot\underline{\xi}$ real $\Rightarrow$ $(\omega^2)^* = \omega^2$

$\Rightarrow$ ω^2 is <u>real</u>

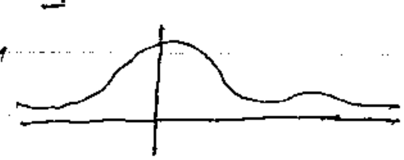
$\therefore$ $\omega^2 > 0 \rightarrow$ stability

$\omega^2 < 0 \rightarrow$ instability, but <u>purely growing</u> $\rightarrow$ no oscillation

Contrast to instabilities with which you should be familiar:

$\rightarrow$ bump-on-tail $\qquad \omega = \omega_k^0 + i\gamma_k$

wave + inverse dissipation $\qquad \gamma \sim \partial f_0/\partial v$



$\downarrow$
carrier

→ two stream $\quad \epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - k v_0)^2}$

- coupling of $\begin{cases} \text{positive energy wave in plasma} \\ \text{negative energy wave in beam} \end{cases}$

- "reactive" counter-part of bump on tail ⟺ can have ω^2 real

⇒ beam + dissipation ⇒ negative energy wave ⊕ dissipation ⇒ growth

$\omega = \omega_r + i\gamma$

- current-driven ion-acoustic

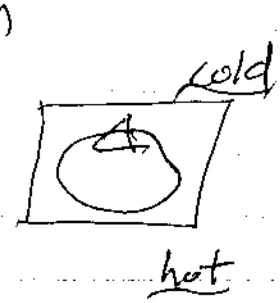
$\omega = \omega_r + i\gamma \qquad \gamma = (\,)\frac{\partial f_e^{(0)}}{\partial v} - (\,)\frac{\partial f_i^{(0)}}{\partial v}$

wave + competition of dissipation and dissipation

⇒ ideal Rayleigh-Benard Convection

$$\omega^2 = -\frac{k_H^2}{k_H^2 + k_V^2}\, g \frac{\partial S}{\partial z}$$

$\partial_z S < 0$



& $g > 0$

of these, ideal MHD instabilities similar in structure to Convection and ω^2 real cases of 2-stream, and different in structure from the others

- In ideal MHD, instability defines structure of eigenfunction, i.e. $\tilde{\underline{\xi}} = \tilde{\underline{\xi}}(\underline{r}, \gamma)$.

N.B. In ideal MHD, only scale in problem is system size $\Leftrightarrow$ boundaries. Contrast Sweet-Parker reconnection ($\Delta/L < 1$), a case of resistive MHD.

proceeding $\Rightarrow$

Since ω^2 real, ω^2 must pass thru $\omega^2 = 0$ as the system evolves from stable to unstable.

- this evolution is called "exchange of stabilities"

- $\Rightarrow$ marginal displacement solves $\underline{F}(\underline{\xi}) = 0$
displacement to perturbed eqlbm.

N.B. $\Rightarrow$ solution of $\underline{F}(\underline{\xi}) = 0$ gives linear stability boundary, in parameter space

ii.) orthogonality

consider two solutions to $-\rho_0 \omega^2 \underline{\xi} = \underline{F}(\underline{\xi})$,

$$-\rho_0 \omega_1^2 \underline{\xi}_1 = \underline{F}(\underline{\xi}_1) \qquad \underline{\times}\ \underline{\xi}_2$$

$$-\rho_0 \omega_2^2 \underline{\xi}_2 = \underline{F}(\underline{\xi}_2) \qquad \underline{\times}\ \underline{\xi}_1$$

$$-(\omega_1^2 - \omega_2^2) \int d^3x\, \rho_0\, \underline{\xi}_1 \cdot \underline{\xi}_2 = \int d^3x \left[\underline{\xi}_2 \cdot \underline{F}(\underline{\xi}_1) - \underline{\xi}_1 \cdot \underline{F}(\underline{\xi}_2) \right]$$

$$= 0 \quad , \text{ by self-adjointness}$$

$$\omega_1^2 \neq \omega_2^2 \Rightarrow \int d^3x\, \rho_0\, \underline{\xi}_1 \cdot \underline{\xi}_2 = 0$$

$\Rightarrow$ orthonormality, with weighting function ρ_0.

The point of all this is that now we can set up a variational quadratic form, ala' beloved Sturm-Liouville theory

$$-\rho\omega^2 \underline{\xi} = \underline{F}(\underline{\xi})$$

and $\otimes \frac{\underline{\xi}}{2}\cdot \Rightarrow$

$$\boxed{\omega^2 = \frac{-\int d^3x\ \underline{\xi}\cdot F(\underline{\xi})/2}{\int \rho_0 \underline{\xi}^2/2}}$$

$$= W_2(\underline{\xi}) \Big/ \int \rho\, \xi^2/2 .$$

$\Rightarrow$ with $K(\underline{\xi}) \equiv \int d^3x\, \rho\, \xi^2/2$, have

$$\boxed{\omega^2 = W_2(\underline{\xi})/K(\underline{\xi})} \rightarrow \left\{ \text{variational, quadratic form} \right.$$

and we know that, since all requirements satisfied, that

$\rightarrow$ any trial ξ plugged into $W_2(\underline{\xi})/K(\underline{\xi})$ yields $\omega^2(\underline{\xi}) > \omega_T^2$

$\hookrightarrow$ the true eigenvalue.

i.e. variational result is always upper bound.

→ so, we know that

- if can find a trial $\underline{\xi}$ such that

$$W_2(\underline{\xi}) < 0$$

- then, configuration is surely unstable

∴ this yields the desired necessary and sufficient condition for instability namely that it be possible to find a $\underline{\xi}$ such that

$$W_2(\underline{\xi}) < 0.$$

hereafter, we write $W_2(\underline{\xi}) = \delta W(\underline{\xi})$,

so the MHD Energy Principle is just:

$$\boxed{\begin{array}{l}\text{instability iff } \exists \text{ well behaved } \underline{\xi} \text{ s/t} \\ \quad \delta W(\underline{\xi}) < 0\end{array}}$$

N.B.

- in physical terms, E.P. $\Rightarrow$ instability if can find a displacement which lowers the energy. Note linear instability $\Longleftrightarrow$ δW to $\mathcal{O}(\xi^2)$ considered

- know $\quad \delta W(\xi) = -\frac{1}{2}\int d^3x \, \underline{\xi} \cdot \underline{F}(\underline{\xi})$

so, now must manipulate δW into physically useful form, i.e. recall

$$F(\xi) = \frac{1}{4\pi}\Big\{ \overbrace{\underline{\nabla} \times \left[\underline{\nabla} \times (\underline{\xi} \times \underline{B_0})\right]\Big\} \times \underline{B_0}}^{\delta \underline{J} \times \underline{B_0} \quad - ①}$$

$$+ \overbrace{\underline{J_0} \times \left[\underline{\nabla} \times (\underline{\xi} \times \underline{B_0})\right]}^{\underline{J_0} \times \delta \underline{B} \quad - ②}$$

$$+ \overbrace{\underline{\nabla}\left[P_0 \, \underline{\nabla} \cdot \underline{\xi} + \underline{\xi} \cdot \underline{\nabla} P_0\right]}^{\nabla \delta p \quad - ③} + \overbrace{\left(\nabla \cdot (\rho_0 \underline{\xi})\right) \underline{\nabla} \phi}^{\delta\rho \nabla\phi \quad - ④}$$

$$= F_① + F_② + F_③ + F_④$$

remember here, all $\underline{B_0}$, $\underline{P_0}$, ρ_0 etc. inhomogeneous, and $\underline{\xi} \cdot \hat{n}$ and $\underline{B} \cdot \hat{n}$ on boundary.

- remains to manipulate $-\int [\underline{\xi}\cdot F(\underline{\xi})/2]\, d^3x$ into "illuminating" form

- key is sign of δW, so seek to extract quadratic terms, as unambiguous.

$\Rightarrow$ let the crank begin!

① $$\delta W_{①} = -\frac{1}{2}\int \underline{\xi}\cdot \underline{F}_0(\underline{\xi})\, d^3x$$

$$= -\frac{1}{2}\int d^3x\, \frac{\underline{\xi}}{4\pi}\cdot \left\{ \left(\underline{\nabla}\times[\nabla\times(\underline{\xi}\times\underline{B}_0)]\right)\times\underline{B}_0 \right\}$$

$$= \frac{1}{8\pi}\int d^3x\, \left(\underline{\nabla}\times[\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)]\right)\cdot\underline{\xi}\times\underline{B}_0$$

$$= \frac{1}{8\pi}\int d^3x\, \nabla\cdot[\nabla\times(\underline{\xi}\times B_0)\times(\underline{\xi}\times\underline{B}_0)]$$
$$+ \frac{1}{8\pi}\int d^3x\, \left(\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)\right)\cdot\left(\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)\right)$$

if $\underline{Q} \equiv \nabla\times(\underline{\xi}\times B_0) = \delta\underline{B}$, from induction

$$\delta W_{①} = \int d^3x\, \frac{\underline{Q}^2}{8\pi} + \frac{1}{8\pi}\int d\underline{S}\cdot\left(\nabla\times(\underline{\xi}\times B_0)\right)\times(\underline{\xi}\times\underline{B}_0)$$

$$\Rightarrow \quad \delta W_{①}\Big|_{\text{surface}} = \frac{-1}{8\pi}\int ds\left[\cancel{\hat{n}}\cdot\underline{B}_0^{\,0}\ \underline{\xi}\cdot\underline{Q} - \cancel{(\hat{n}\cdot\underline{\xi})}^{\,0}\ \underline{B}_0\cdot\underline{Q}\right]$$

$$\delta W_{①} = \int d^3x\ \frac{Q^2}{8\pi}$$

$$\delta W_{②} = -\frac{1}{2}\int d^3x\ \underline{\xi}\cdot\underline{J}_0\times\left[\nabla\times(\underline{\xi}\times\underline{B}_0)\right]$$

$$= -\frac{1}{2}\int d^3x\ \underline{\xi}\cdot\left(\underline{J}_0\times\underline{Q}\right)$$

$$= +\frac{1}{2}\int d^3x\ \underline{J}_0\cdot\left(\underline{\xi}\times\underline{Q}\right)$$

$$\delta W_{③} = -\frac{1}{2}\int d^3x\ \underline{\xi}\cdot\underline{\nabla}\left[p_0\,\underline{\nabla}\cdot\underline{\xi} + \underline{\xi}\cdot\underline{\nabla}p_0\right]$$

ibp $\underline{\xi}\cdot\hat{n} = 0$ on boundary

$$\Rightarrow \delta W_{③} = \int\frac{d^3x}{2}\left[p_0\left(\underline{\nabla}\cdot\underline{\xi}\right)^2 + \left(\underline{\nabla}\cdot\underline{\xi}\right)\underline{\xi}\cdot\underline{\nabla}p_0\right]$$

& last but not least...

$$+ \underline{\xi} \cdot \underline{\nabla}\phi \, \delta\rho$$

$$\delta W_{④} = \int \frac{d^3x}{2} \cancel{\underline{\xi} \cdot \left[\nabla \cdot (\rho_0 \underline{\xi})\right] \underline{\nabla}\phi}$$

$$= -\frac{1}{2} \int d^3x \, (\underline{\xi} \cdot \underline{\nabla}\phi) \, \nabla \cdot (\rho_0 \underline{\xi})$$

so, putting the whole mess together

$$\boxed{\begin{aligned} \delta W = \frac{1}{2} \int d^3x \Big\{ & \overset{①}{\frac{\underline{Q}^2}{4\pi}} + \overset{②}{\underline{J}_0(\underline{x}) \cdot (\underline{\xi} \times \underline{Q})} \\ & + \overset{③}{\gamma P_0(\underline{x}) (\underline{\nabla} \cdot \underline{\xi})^2} + \overset{④}{(\underline{\xi} \cdot \underline{\nabla} P_0(\underline{x})) \, \underline{\nabla} \cdot \underline{\xi}} - \overset{⑤}{(\underline{\xi} \cdot \underline{\nabla}\phi) \, \nabla \cdot (\rho_0 \underline{\xi})} \Big\} \\ \underline{Q} = \nabla \times (\underline{\xi} \times \underline{B}_0) & \end{aligned}}$$

note: general characteristics

- ① → > 0 → field line bending
  ③ → > 0 → compression
  } → always stabilizing $\delta W > 0$.

- free energy sources:
  - $\nabla \rho_0(\underline{x})$ in ⑤ ⟶ density gradient
  - $\underline{J}_0(\underline{x})$ in ② ⟶ current profile
  - $\underline{\nabla} P_0(\underline{x})$ in ④ ⟶ pressure gradient
  - gravity and ρ_0 in ⑤

⇒ can make $\delta W < 0$, for certain profiles and $\underline{\xi}$ ⇒ free energy sources for instability.

Note:

→ δW is imprecise

→ δW does not reveal much about growth rates

but

→ very useful for simple quick assessment of stability

→ can elucidate

- complex problem
- problem in which info re: equilibrium not precise.

$\therefore$ further developments in theory remain, but better to consider some examples

$\Rightarrow$

iii) Convection and Interchange Instabilities
$\rightarrow$ A Simple Application of the Energy Principle

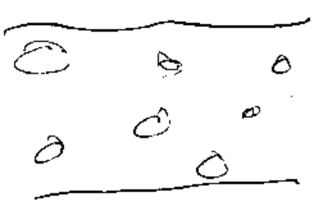
consider 4 related examples:

a.) Convection and the Schwarzschild Criterion
b.) Rayleigh-Taylor Instability
c.) Interchange Instability
d.) Interchange Without Gravity

i.) Schwarzschild Criterion and Convection

i.e. stellar atmosphere

$$\left(\rho g = \frac{dp}{dz} \right)$$

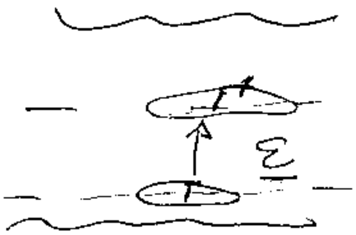


$z \uparrow \quad \frac{d\rho}{dz} < 0, \ \frac{dP}{dz} < 0$

and $\quad P \rho^{-\gamma} = \text{const.}$ $\quad$ (basic means of heat transport)

$\therefore$ basic idea of convection, consider a virtual displacement of a slug/blob of gas upward

$\Rightarrow$ physical argument



$-z_2$ $\stackrel{\text{if}}{=}$ $\rho_1' > \rho_2 \rightarrow$ blob sinks
system stable

z_1

$\rho_1' < \rho_2 \rightarrow$ blob rises,
system unstable

for infinitesimal displacement, $\varepsilon \sim \Delta z \Rightarrow$

$$\rho_2 = \rho_1 + \frac{d\rho_1}{dz} \Delta z$$

For ρ_1'; $\rightarrow$ system is isentropic $\Rightarrow$
$p\rho^{-\gamma} = \text{const}$ applies

$\rightarrow$ displaced blob (i.e. ①′) comes
to rapid pressure equilibration
with surroundings

i.e. $\boxed{\frac{\Delta z}{c_s} \ll \tau_{rise}}$ $\leftrightarrow$ $\gamma < k c_s$
~ 'nearly incompressible'

$$\therefore \quad p_1' = p_1 + \Delta z \frac{dp_1}{dz} = p_2$$

isentropy

$$p_1 \rho_1^{-\gamma} = p_1' \rho_1'^{-\gamma}$$

$$\Rightarrow \quad P_1 \rho_1^{-\gamma} = \left(P_1 + \Delta z \frac{dP_1}{dz}\right) \rho_1'^{-\gamma}$$

$$\Rightarrow \quad \left(\frac{\rho_1'}{\rho_1}\right)^{\gamma} = 1 + \frac{\Delta z}{P_1} \frac{dP_1}{dz}$$

$$\frac{\rho_1'}{\rho_1} = \left(1 + \frac{\Delta z}{P_1} \frac{dP_1}{dz}\right)^{1/\gamma} \cong 1 + \frac{\Delta z}{\gamma P_1} \frac{dP_1}{dz}$$

$$\frac{\rho_2}{\rho_1} = 1 + \frac{1}{\rho_1} \Delta z \frac{d\rho_1}{dz}$$

$\Rightarrow$ buoyant blob if:

$$\frac{\rho_1'}{\rho_1} < \frac{\rho_2}{\rho_1} \Rightarrow \frac{\Delta z}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz} < \frac{\Delta z}{\rho_1} \frac{d\rho_1}{dz}$$

$$\Rightarrow \boxed{\frac{1}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz} < \frac{1}{\rho_1} \frac{d\rho_1}{dz}}$$

or, as both gradients negative

$$\boxed{\frac{1}{\gamma} \left| \frac{1}{P_1} \frac{dP_1}{dz} \right| > \frac{1}{\rho_1} \left| \frac{d\rho_1}{dz} \right|}$$

Schwarzchild criterion for Convective Instability

and as $S \equiv \ln\left(P \rho^{-\gamma}\right)$

$$\frac{dS}{dz} = \frac{1}{p}\frac{dp}{dz} - \frac{\gamma}{\rho}\frac{d\rho}{dz}$$

$\Rightarrow$ blob buoyant if $\dfrac{dS}{dz} < 0$ $\rightarrow$ "superadiabatically stratified"

sinks/restored if $\dfrac{dS}{dz} > 0 \rightarrow$ "subadiabatically stratified"

Marginal $dS/dz = 0$ $\rightarrow$ adiabatically stratified

Note: $\rightarrow$ Schwarzschild instability criterion $\leftrightarrow$ answers "is free energy available, locally" $\leftrightarrow$ ideal

$\rightarrow$ Rayleigh # criterion $\Rightarrow$ $Ra > Ra)_{crit}$
$\Rightarrow$ does free energy overcome dissipation?

Now, what does δW say?

Recall: $$\delta W = \frac{1}{2}\int d^3x \left[\frac{Q^2}{4\pi} + \gamma P (\underline{\nabla}\cdot\underline{\xi})^2 + \hat{\underline{J}}_0\cdot(\underline{\xi}\times\underline{Q}) + (\underline{\xi}\cdot\underline{\nabla}P_0)(\underline{\nabla}\cdot\underline{\xi}) - (\underline{\xi}\cdot\underline{\nabla}\phi)\nabla\cdot(\rho_0\underline{\xi}) \right]$$

pure hydro $\rightarrow$ $\underline{Q} = 0$, $\underline{j}_0 = 0$

$\dfrac{dp}{dz} = \rho g$ $\rightarrow$ hydrostatic equilibrium
$\underline{\nabla}P = \cancel{\underline{J}\times\underline{B}} + \rho\underline{g}$

$\underline{g} = \nabla\phi$, $\underline{g}$ downward

$\therefore$

$$2\delta W = \int d^3x \Big[\gamma p\, (\nabla\cdot\underline{\xi})^2 + (\underline{\xi}\cdot\nabla p)(\nabla\cdot\underline{\xi}) + (\underline{\xi}\cdot\underline{g})\left(\underline{\xi}\cdot\nabla\rho_0 + \rho_0\nabla\cdot\underline{\xi}\right) \Big]$$

$$= \int d^3x \Big[\gamma p\, (\nabla\cdot\underline{\xi})^2 + (\nabla\cdot\underline{\xi})\left(\underline{\xi}\cdot(\nabla p + \underline{g}\rho_0)\right) + (\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\nabla\rho_0)$$

but $\nabla p = \rho\,\underline{g}$ (eqbrm condition) $\Rightarrow$

$$2\delta W = \int d^3x \Big[\gamma p\left((\nabla\cdot\underline{\xi})^2 + 2\frac{(\nabla\cdot\underline{\xi})(\underline{\xi}\cdot\nabla p)}{\gamma p} + \left(\frac{\underline{\xi}\cdot\nabla p}{\gamma p}\right)^2\right) - \gamma p\left(\frac{\underline{\xi}\cdot\nabla p}{\gamma p}\right)^2 + (\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\nabla\rho_0) \Big]$$

$$= \int d^3x \Big[\gamma p\left(\nabla\cdot\underline{\xi} + \frac{\underline{\xi}\cdot\nabla p}{\gamma p}\right)^2 - \frac{(\underline{\xi}\cdot\nabla p)^2}{\gamma p} + (\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\nabla\rho_0) \Big]$$

$$= \int d^3x \Big[\gamma p\left(\nabla\cdot\underline{\xi} + \frac{\underline{\xi}\cdot\nabla p}{\gamma p}\right)^2 - \underline{\xi}\cdot\nabla p\left(\frac{\underline{\xi}\nabla p}{\gamma p} - \frac{\underline{\xi}\cdot\nabla\rho_0}{\rho_0}\right) \Big]$$

where used equilibrium condition again, so

$\Rightarrow$

$$2\delta W = \int d^3x \left[\gamma P \left(\underline{\nabla}\cdot\underline{\xi} + \frac{\underline{\xi}\cdot\underline{\nabla}P}{\gamma P} \right)^2 - \frac{\underline{\xi}\cdot\underline{\nabla}P}{\gamma}\, \underline{\xi}\cdot\underline{\nabla} \ln\left(P\rho^{-\gamma}\right) \right]$$

Now, object is to
- $\rightarrow$ explore possible displacements to see if $\delta W < 0$ possible
- $\rightarrow$ uncover any general condition

Now, expect $\underline{\xi}$ to have form:

$$\underline{\xi} = \mathrm{re}\left[\underline{\hat{\xi}}(z)\, e^{ikx}\right] \qquad \text{(must be real!)}$$

so can <u>choose</u> $\underline{\nabla}\cdot\underline{\xi} = -\dfrac{\underline{\xi}\cdot\underline{\nabla}P}{\gamma P}$

$\rightarrow$ equivalent to setting a relation between ξ_x, ξ_z.

$$\rightarrow \quad \underline{\nabla}\cdot\underline{\xi} \sim \frac{\xi}{\gamma P}\frac{dP}{dz} \sim \frac{\xi}{\gamma L_P}$$

$\mapsto$ pressure scale height

so $\dfrac{|\nabla\cdot\xi|}{|\xi|} \sim 1/L_P \rightarrow$ "weakly compressible", in accord with physical argument contrast $\frac{|\nabla \cdot \xi|}{|\xi|} \sim |k|$ → "strongly compressible" limit

so $2\delta W = -\int d^3x \left[\frac{\underline{\xi} \cdot \nabla P}{\gamma} \; \underline{\xi} \cdot \nabla \ln \left(P \rho^{-\gamma} \right) \right]$

$\frac{dP}{dz} \hat{z} = \nabla P$ and $\frac{dP}{dz} < 0$ ⇒

if have any range of z over which

$$\frac{d}{dz} \ln (P\rho^{-\gamma}) < 0$$

⇒ have $\underline{\xi} \neq 0$ there, and $\delta W < 0$

⇒ instability, with criterion/condition that

$$\boxed{\frac{d \ln (P\rho^{-\gamma})}{dz} < 0}$$ → Schwarzschild Condition recovered

Now, can go furthur, and ask what is effect of magnetic field?

i.e. — consider $\underline{B} = B_0 \hat{x}$

then

$$\delta W = \underbrace{\delta W_0}_{\text{what we have}} + \int d^3x \, \frac{Q^2}{8\pi}$$

$$\underline{Q} = \underline{\nabla} \times (\underline{\xi} \times \underline{B}_0)$$

$$\underline{Q} = \underline{B}_0 \cdot \underline{\nabla}\, \underline{\xi} - \overbrace{\cancel{\underline{\xi} \cdot \underline{\nabla}\, \underline{B}_0}}^{0 \text{ (homogeneous)}} - \underline{B}_0 \, \underline{\nabla} \cdot \underline{\xi}$$

Now, to minimize δW, $\underline{B}_0 \cdot \underline{\nabla}\, \underline{\xi} = 0$

→ flute displacement $k_\parallel = 0$

→ no bending energy expended

i.e. $Q = -B_0 \, \underline{\nabla} \cdot \underline{\xi}$

$$\delta W = \delta W_0 + \int d^3x \, \frac{B_0^2}{8\pi} (\underline{\nabla} \cdot \underline{\xi})^2$$

but from before have, $\underline{\nabla} \cdot \underline{\xi} = -\dfrac{\underline{\xi} \cdot \underline{\nabla} p}{\gamma p}$

$$\delta W = \int d^3x \left[\frac{B_0^2}{8\pi} \left(\frac{\underline{\xi} \cdot \nabla p}{\gamma p} \right)^2 - \left(\frac{\underline{\xi} \cdot \underline{\nabla} p}{\gamma} \right) \frac{\underline{\xi}}{2} \cdot \nabla \ln (p \rho^{-\gamma}) \right]$$

$$\delta W \sim \int d^3x \left[P_{mag} \frac{\xi^2}{\gamma^2 L_p^2} - \frac{P_{th}}{\gamma L_p} \xi^2 \left| \frac{ds}{dz} \right| \right]$$

$$\delta W < 0 \quad \text{if} \quad \left| \frac{ds}{dz} \right| > \frac{P_{mag}}{P_{th}\, \gamma L_p}$$

$$\Rightarrow \quad \frac{ds}{dz} < \frac{1}{\gamma \beta} \left(\frac{dP}{P\, dz} \right)$$

$\therefore$ indicates $\rightarrow$ magnetic field stabilizing
$\rightarrow$ need critical entropy gradient $\sim \frac{1}{\beta L_p}$ for instability.

Moral of the story:

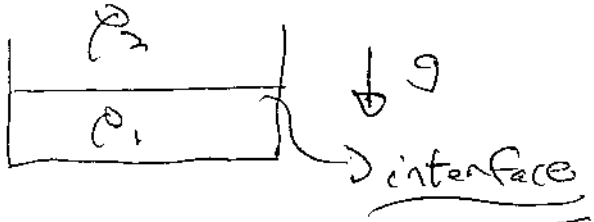
- $\rightarrow$ energy principle recovers essential physical criterion (Schwarzschild)
- $\rightarrow$ enables simple, quick albeit imprecise insights into more complicated stability problems.

$$\frac{d}{dz}\left(p\rho^{-\gamma}\right) = \frac{d}{dz}\left(T\rho^{-(\gamma-1)}\right) < 0$$

$$= \left(\frac{d}{dz}T\right)\rho^{-(\gamma-1)} - (\gamma-1)\rho^{-\gamma}\frac{d\rho}{dz} < 0$$

$$\frac{dT}{dz} < (\gamma-1)\frac{1}{\rho}\frac{d\rho}{dz}$$

b.) <u>Rayleigh-Taylor Instability</u> → critical to implosions (ICF)



$\downarrow g$ $\qquad \rho_2 > \rho_1$ $\qquad$ $p = 0$ (cold) (as will $\underline{\nabla} \cdot \underline{V} = 0$)

→ while nominally at equilibrium, configuration is unstable (heavy "falls" into light)

→ 

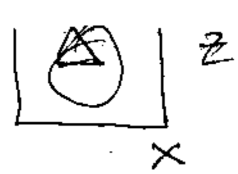
→ ripples, "spike and bubble"

$$\gamma^2 = |kg| \frac{(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)}$$

→ here $\underline{\nabla} \cdot \underline{V} = 0$

→ if continuous profile $\qquad \downarrow g \qquad \nabla \rho \uparrow$

$$\frac{\partial \tilde{V}}{\partial t} = -\frac{\nabla \tilde{p}}{\rho_0} - g \frac{\tilde{\rho}}{\rho_0} \hat{z} \qquad g > 0$$



$$\frac{\partial}{\partial t} (\underline{\nabla} \times \underline{\tilde{V}}) \cdot \hat{y} = 0 - g \nabla \times \left(\frac{\tilde{\rho}}{\rho_0} \hat{z} \right)$$

$$\underline{V} = -\partial_z \phi \, \hat{x} + \partial_x \phi \, \hat{z}$$

$$\underline{\nabla} \cdot \underline{V} = 0$$

$$-\frac{\partial}{\partial t}\nabla^2\phi = g\,\partial_x\left(\frac{\tilde{\rho}}{\rho_0}\right)$$

$$\frac{\partial\tilde{\rho}}{\partial t} = -\partial_x\tilde{\phi}\,\frac{d\rho_0}{dz} \Rightarrow \omega^2 = -\frac{k_x^2 g}{k^2 L_\rho}$$

$$\boxed{\gamma^2 = \frac{k_x^2}{k^2}\, g/L_\rho} \qquad \begin{array}{l} g>0 \\ 1/L_\rho > 0 \end{array}$$

interchange structure

Now, what would δW say?

$$\delta W = \frac{1}{2}\int d^3x\left[\frac{\underline{Q}^2}{4\pi} + \gamma P(\underline{\nabla}\cdot\underline{\xi})^2 + \underline{J}_0\cdot(\underline{\xi}\times\underline{Q}) + (\underline{\xi}\cdot\underline{\nabla}P_0)(\underline{\nabla}\cdot\underline{\xi}) - (\underline{\xi}\cdot\underline{\nabla}\phi)(\nabla\cdot(\rho_0\underline{\xi}))\right]$$

$$\underline{Q}=0,\ \underline{j}=0,\ \nabla\underline{P}=0,\ \underline{\nabla}\cdot\underline{\xi}=0$$

$$\delta W = \int d^3x\left[-(\underline{\xi}\cdot\underline{\nabla}\phi)(\rho_0\underline{\nabla}\cdot\underline{\xi} + \underline{\xi}\cdot\underline{\nabla}\rho_0)\right]$$

$$= \int d^3x\left[+(\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\underline{\nabla}\rho_0)\right]$$

$$\delta W = \int\frac{d^3x}{2}\left[(\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\underline{\nabla}\rho_0)\right]$$

$$\delta W = \int \frac{d^3x}{2}\left[(\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\underline{\nabla}\rho_0)\right]$$

$g<0$ so if $\underline{\nabla}\rho_0 > 0$ $(d\rho_0/dz > 0)$ anywhere

$\Rightarrow \delta W < 0 \rightarrow$ instability

Now, if equilibrium hydrostatic:

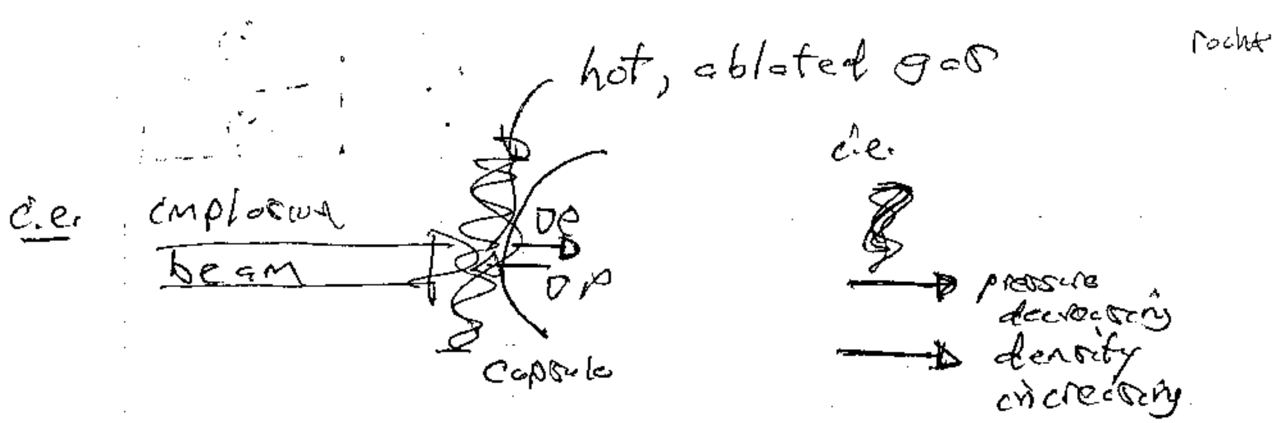
$\underline{\nabla} p = \rho \underline{g}$, $\Rightarrow$

$$\boxed{\delta W = \int \frac{d^3x}{2}\left[(\underline{\xi}\cdot\underline{\nabla}p)\left(\frac{\underline{\xi}\cdot\underline{\nabla}\rho_0}{\rho_0}\right)\right]}$$

$\Rightarrow$ Rayleigh Taylor instability will result whenever $(\nabla p)(\nabla\rho) < 0$

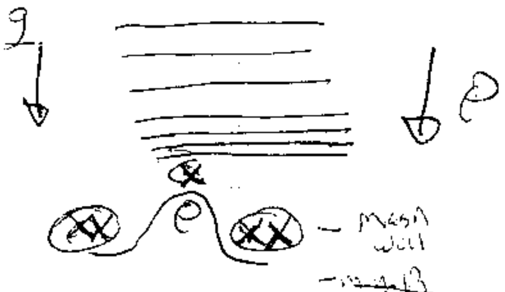
$\rightarrow$ pressure density gradients opposite

i.e. "heavy supported by light" (i.e. pressure highest at bottom)



iii) Interchange Instability (basic confinement consideration)

→ consider plasma confined by magnetic pressure gradient



$$\underline{\nabla} p = \underline{J} \times \underline{B} + \rho \underline{g}$$

str. lines

$$\frac{dp}{dz} = -\underline{\nabla}\left(\frac{B^2}{8\pi}\right) + \cancel{\frac{\underline{B}\cdot\underline{\nabla}\underline{B}}{4\pi}} + \rho \underline{g}$$

$\beta \ll 1$

$$\boxed{\underline{\nabla}\left(\frac{B^2}{8\pi}\right) = \rho \underline{g}}$$

$\underline{g} = -g\,\hat{z}$

$p \rightarrow 0$

equilibrium

→
$$\delta W = \int d^3x \left[\frac{\underline{Q}^2}{8\pi} + (\underline{\nabla}\cdot\underline{\xi})^2 \gamma p_0 + \underline{j}_0 \cdot (\underline{\xi} \times \underline{Q}) + (\underline{\xi}\cdot\underline{\nabla} p_0)(\underline{\nabla}\cdot\underline{\xi}) - (\underline{\xi}\cdot\underline{\nabla}\phi)\,\underline{\nabla}\cdot(\rho_0 \underline{\xi}) \right]$$

$j_0 = 0$

$p_0 = 0$

$$\delta W = \int d^3x \left[\frac{\underline{Q}^2}{8\pi} + (\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\underline{\nabla}\rho_0 + \rho_0 \underline{\nabla}\cdot\underline{\xi}) \right]$$

Here, must address $\underline{Q}$,

$$\underline{Q} = \underline{B_0}\cdot\underline{\nabla}\,\underline{\xi} - \underline{\xi}\cdot\underline{\nabla}\,\underline{B} - \underline{B_0}\,\underline{\nabla}\cdot\underline{\xi}$$

Now, can have $\underline{Q} = 0$ if:

$\rightarrow$ $\underline{B_0}\cdot\underline{\nabla}\,\xi = 0$ i.e. $\underline{\xi}$ constant along $\underline{B_0}$

$\Rightarrow k_{\parallel} = 0$

and

$$\rightarrow \quad \underline{\nabla}\cdot\underline{\xi} = -\frac{\underline{\xi}\cdot\underline{\nabla}B_0}{B_0}$$

$\therefore$

$$\delta W = \int d^3x \left[(\underline{\xi}\cdot\underline{g})\,\rho_0 \left(\frac{\underline{\xi}\cdot\underline{\nabla}\rho_0}{\rho_0} - \frac{\underline{\xi}\cdot\underline{\nabla}B_0}{B_0} \right) \right.$$

$$= \int d^3x \left[(\underline{\xi}\cdot\underline{g}\rho_0)\ \underline{\xi}\cdot\underline{\nabla}\ln(\rho/B) \right]$$

$g < 0 \Rightarrow$ if $\nabla \ln(\rho/B) > 0$ anywhere

$\therefore$ instability there!

Now:

→ obvious parallel to Rayleigh - Taylor is

$\nabla \rho > 0 \iff \nabla \ln(P/B) > 0$

→ as $k_{\parallel} = 0$, field lines not bent

so can think of instability motion as interchange of flux tubes

Key question: Does interchange lower/raise potential energy?

interchange conserves magnetic flux equiv

$\Phi_2 = \int B_2 \, da = B_2 A_2$

$\Phi_1 = \int B_1 \, da = B_1 A_1$

$M_2 = \left(\frac{\rho}{B}\right)_2 \Phi_2$

$M_1 = \left(\frac{\rho}{B}\right)_1 \Phi_1$

$M \Leftrightarrow$ m/length

$M = \rho A$

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but $\phi_1 = \phi_2 \Rightarrow$

$$M_{2,1} = (\rho/B)_{2,1}\, \Phi$$

so $\nabla M \gtrless 0 \Rightarrow \nabla(\rho/B) \gtrless 0$

$\Rightarrow$ if ρ/B increases interchange will liberate gravitational potential energy, i.e. instability, a la' R-T.

$\rightarrow$ why care?:

- (interchange instability) severely degrades plasma confinement

- curing interchange stability is key element in device design $\rightarrow$ "minimum-B" magnetic well

iv.) Interchange without Gravity

- β limit
- expansion free energy?

- in the context of magnetic confinement, "g" is a crutch, to represent curved field lines

— i.e. 

$$\underline{g} = \frac{V^2}{R_c} \rightarrow \underline{g}_{eff}$$

[centrifugal acceleration on particle]

— natural to investigate interchanges without "$\underline{g}$" $\Rightarrow$ pressure gradient drive (expansion free energy)

— now

$$\delta W = \int d^3x \left[\frac{Q^2}{8\pi} + \gamma p (\underline{\nabla}\cdot\underline{\xi})^2 + \underline{\xi}\cdot\underline{\nabla} p\,(\underline{\nabla}\cdot\underline{\xi}) + \underline{j}\cdot\underline{\xi}\times\underline{Q} \right]$$

Now, $\underline{Q} = 0$ $\rightarrow$ avoid bending, etc.

$$\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0) = 0$$

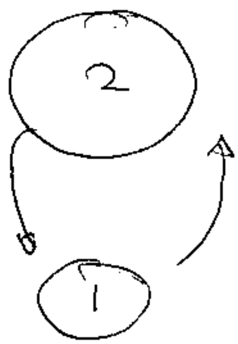
$$\Rightarrow \quad \underline{\xi}\times\underline{B}_0 = \nabla\phi$$

$\hookrightarrow$ some scalar potential

and $\underline{B}\cdot\underline{\nabla}\phi = 0$ $\Rightarrow$ ϕ constant along lines of force ...

and can formulate dW in terms ϕ, or....

$\Rightarrow$ consider interchange, with flux conservation



$$\Phi_1 = \Phi_2$$

Does interchange raise or lower energy?

ΔE = [final energy of ①] − [initial energy of ①]
+ [final energy of ②] − [initial energy of ②]

where interchange

a) "puts" ① into ② slot
"puts" ② into ① slot

b) Keeps $P\rho^{-\gamma} = P V^{\gamma} = \text{const}$

$V \equiv$ Volume of flux tube

$\Rightarrow$ final energy of ① $\rightarrow \underbrace{(\text{new } P)}_{P_1'} V_2 / (\gamma - 1)$

final energy of ② $\rightarrow \underbrace{(\text{new } P)}_{P_2'} V_1 / \gamma - 1$

so ...

$$\Delta E = \Delta W = \frac{1}{(\gamma-1)}\left[\left(p_1' V_2 - p_1 V_1\right) + \left(p_2' V_1 - p_2 V_2\right)\right]$$

and
$$\left.\begin{aligned} p_1' V_2^\gamma &= p_1 V_1^\gamma \\ p_2' V_1^\gamma &= p_2 V_2^\gamma \end{aligned}\right\}$$
from eqn. state
$p' \equiv$ pressures of displaced flux tubes
(argument akin to Schwarzschild)

$\Rightarrow$

$$(\gamma-1)\,\Delta W = \left\{ p_1\left[\left(\frac{V_1}{V_2}\right)^\gamma V_2 - V_1\right] + p_2\left[\left(\frac{V_2}{V_1}\right)^\gamma V_1 - V_2\right]\right\}$$

$$V_2 = V_1 + \delta V$$
$$p_2 = p_1 + \delta p$$

Shiss

$$(\Delta W)(\gamma-1) = \left\{ p_1\left[\left(\frac{V_1}{V_1+\delta V}\right)^\gamma (V_1+\delta V) - V_1\right] + (p_1+\delta p)\left[\left(\frac{V_1+\delta V}{V_1}\right)^\gamma V_1 - (V_1+\delta V)\right]\right\}$$

$$(\gamma-1)\,\Delta W = \left\{ P_1 V_1 \left[\left(1+\frac{\delta V}{V}\right)^{-(\gamma-1)} - 1 \right] + P_1 V_1 \left(1+\frac{\delta p}{p}\right)\left[\left(1+\frac{\delta V}{V}\right)^{\gamma} - \left(1+\frac{\delta V}{V}\right) \right] \right\}$$

$$= P_1 V_1 \left\{ \left[\cancel{1} - (\gamma-1)\frac{\delta V}{V} + \frac{(\gamma-1)(\gamma)}{2}\left(\frac{\delta V}{V}\right)^2 - \cancel{1} \right] + \left(1+\frac{\delta p}{p}\right)\left[\cancel{1} + \gamma\frac{\delta V}{V} + \gamma\frac{(\gamma-1)}{2}\left(\frac{\delta V}{V}\right)^2 - \cancel{1} - \frac{\delta V}{V} \right] \right\}$$

$$= P_1 V_1 \left\{ \cancel{-(\gamma-1)\frac{\delta V}{V}} + \frac{(\gamma-1)(\gamma)}{2}\left(\frac{\delta V}{V}\right)^2 + \cancel{\gamma\frac{\delta V}{V}} - \cancel{\frac{\delta V}{V}} + \frac{\delta p}{p}(\gamma-1)\frac{\delta V}{V} + \gamma\frac{(\gamma-1)}{2}\left(\frac{\delta V}{V}\right)^2 \right\}$$

$\therefore$

$$\boxed{\frac{\Delta W}{P_1 V_1} = \underbrace{\gamma\left(\frac{\delta V}{V}\right)^2}_{>0} + \underbrace{\frac{\delta p}{p}\frac{\delta V}{V}}_{>0 \text{ or } <0}}$$

$\rightarrow$ generic expression for interchange δW

Review
- R-G
- R-T
- Int I
- Int II - exp. free

$$\Phi_1 = \Phi_2 \quad Q = 0 \to \text{curvature} (?)$$

$$\frac{\delta W}{p V} = \gamma \left(\frac{\delta V}{V}\right)^2 + \frac{\delta p}{p}\frac{\delta V}{V}$$

>0, <0 → curv → criterion stab.

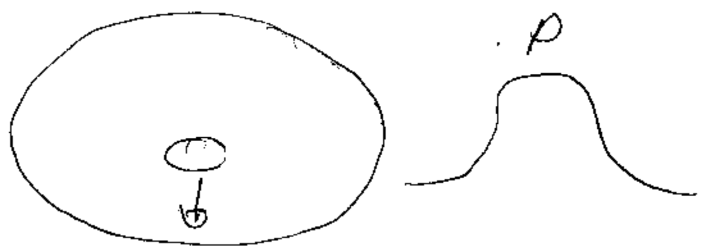
Clearly,

$$\frac{\delta V}{V} \sim (\nabla \cdot \xi) \quad , \quad \frac{\delta p}{p} \sim \quad \xi \cdot \nabla p$$

and

→ expansion free energy relaxation ⇒

$\underline{\delta p < 0}$ → i.e.



pressure higher in center so $\delta p < 0$ ⇒ relaxation occurs.

∴ Key is sign $\frac{\delta V}{V}$
→ > 0 → instability (maybe)
→ < 0 → stability (sure)

→ Now, for flute perturbation $(k_{\parallel} = 0)$

$$V = \int S \, dl$$

$S \equiv$ cross-sectional area of tube.

but $\Phi = B(l)\, S(l) = \text{const}$

$\Rightarrow$

$$V = \Phi \int \frac{dl}{B} \quad \Rightarrow \quad \frac{\delta V}{V} < 0$$

$\Rightarrow$ $\boxed{\delta \int \frac{dl}{B} < 0}$ $\rightarrow$ condition for interchange stability

$$\underset{<0}{\delta p}\ \underset{<0}{\delta V} > 0$$

- content of criterion is that configuration should have a minimum in B in the core, to confine pressure

i.e.

then stable if:

$$\delta \int \frac{dl}{B} < 0$$

$\Rightarrow$ "minimum B" criterion for stability.

magnetic well

→ if define ψ → label of surface enclosing const flux Φ

∴ $V(\psi) \equiv$ volume enclosed by flux surface

$p(\psi) \equiv$ pressure enclosed

$dp/d\psi < 0 \Rightarrow$ need $\dfrac{d^2V}{d\psi^2} > 0$ ✓ $V' < 0$

⟷ minimum-B

→ Can re-write instability criterion

$$\delta W = p_1\, \delta V \left(\gamma \frac{\delta V}{V_1} + \frac{\delta p}{p_1} \right)$$

$$= \boxed{p_1\, \delta V \left[\delta \ln (p V^{\gamma}) \right]}$$

so $\delta(pV^{\gamma}) < 0$ → inst. (cf. Schwarzschild)

Also, if tube ① has flux ψ_0 then

$$v = u\,\psi$$

$\Rightarrow$

$$\frac{\delta w}{\psi} = p\,\delta u \;\frac{\delta(p u^{\gamma})}{p u^{\gamma}} < 0$$

→ What does it Mean?

$$\underset{\text{volume}}{V} = \int dl\, A = \Phi \int \frac{dl}{B}$$

now $\underline{\nabla} p$ → "expansion free energy"

$\delta V > 0 \Rightarrow \delta \int \frac{dl}{B} > 0$ → fluid element expands
⇒ tends reduce W_p

$\delta V < 0 \Rightarrow \delta \int \frac{dl}{B} < 0$ → fluid element compresses
⇒ tends increase W_p

$\delta V > 0$ → 'maximum B' hill
$\delta V < 0$ → 'minimum B' well

Can then define:

Want min. for min. B ⇒ −S.T.I.

$$\underset{\text{potential energy of tube}}{E_p} = -p\,\mathcal{U} \quad , \quad \mathcal{U} = -\oint \frac{dl}{B}$$

(i.e. for sign convention)

∴ → can argue tube tends to move in direction of lower $\mathcal{U}$.
→ equilibrium for $p = p(\mathcal{U})$

then, not surprisingly, can develop parallel between convection and interchange

i.e.

Convection	Interchange
gravitational potential energy	$E_p \to$ expansion energy
blob	flux tube
displace blob	displace tube
$\rho' < \rho_{\text{ambient}}$ $\leftrightarrow$ buoyant rise	$\frac{\delta V}{V} > 0$ $\leftrightarrow$ expansion continues (with $dp/dU < 0$) (squeezed out)
adiabatic profile $\frac{\nabla p}{p} = \gamma \frac{\nabla \rho}{\rho}$	adiabatic displacement $\Delta E_p = \Delta(-pU^{\gamma}) = (U^{\gamma} \frac{dp}{dU} + \gamma U^{\gamma} \frac{p}{U})\delta U =$ $-\gamma p \frac{\delta U}{U} = \frac{dp}{dU} \delta U$
Schwarzschild Criterion $\frac{\nabla p}{p} < \gamma \frac{\nabla \rho}{\rho}$ for instability	Interchange Criterion $\frac{dp}{dU} \delta U < -\gamma p \frac{\delta U}{U}$ $\Rightarrow \boxed{\left\lvert \frac{dp}{dU} \right\rvert > -\frac{\gamma p}{U}}$ ∇p sufficiently ... overcome compr.</br>

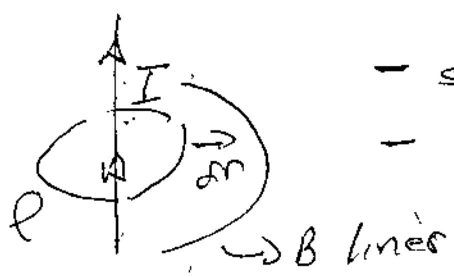
$\therefore$ for instability : $\left|\underbrace{\frac{dp}{du}}_{\text{change from relaxation}}\right| > \underbrace{-\frac{\gamma p}{u}}_{\text{adiabatic pressure change}}$

for stability, need:

$$\boxed{\left|\frac{dp}{du}\right| < \frac{\gamma p}{|u|}}$$

→ Consider some configurations (magnetic)

a.) single wire



- stability to displacement δr
- $\delta p/p$ limit ?

now $\delta \int \frac{d\ell}{B}$ $\qquad d\ell = 2\pi r$

$B = 2I/r$

$d\ell / B \sim \frac{\pi}{I} r^2$ → "wire is not "minimum $-B$" !

i.e. actually maximal ?

→ will have a δp crit.

for δp limit: $\quad \frac{dp}{dU} < \frac{\gamma p}{|U|}$

$$U \equiv -\int \frac{d\ell}{B} \sim -r^2 \frac{\pi}{I}$$

$$\frac{dp}{dU} = \frac{dp}{dr}\frac{dr}{dU}$$

$\boxed{U \text{ scalar} \Rightarrow I \text{ cancels}}$

$$= \left|\frac{dp}{dr}\right|\left(\frac{1}{2r}\right)\left(\frac{I}{\pi}\right) \Rightarrow \left|\frac{1}{p}\frac{dp}{dr}\right| < \frac{\gamma(2r)}{r^2}$$

$$\therefore \left|\frac{1}{p}\frac{dp}{dr}\right| < \frac{2\gamma}{r} \Rightarrow \boxed{\left|\frac{d\ln p}{d\ln r}\right| < 2\gamma}$$

$\Rightarrow$ imposes limit on pressure gradient for interchange stability. $\Rightarrow$ "β limits"

can approach point dipole similarly $\rightarrow$ i.e. earth.

i.e. $B \sim 1/r^3$

$d\ell \sim r$ $\quad \Rightarrow \quad U \sim -r^4$

HW $\rightarrow$ show

similar reasoning $\Rightarrow -\frac{d\ln p}{d\ln r} < 4\gamma$

Recall, established interchange stability criterion:

$$\frac{\delta W}{W_0} = \gamma\left(\frac{\delta V}{V_0}\right)^2 + \left(\frac{\delta p}{p_0}\right)\left(\frac{\delta V}{V_0}\right)$$

compression in δW — Δ[expansion free energy] in δW

so $\frac{\delta W}{W_0} > 0$ if: $\frac{\delta V}{V_0} < 0$ (for non-trivial case where $\frac{\delta p}{p_0} < 0$)

$$\Phi = B\, dA(\ell)$$

$$\Rightarrow \quad \delta V = \delta \int \frac{d\ell}{B}, \quad \text{as } \Phi = \text{const}$$

$\therefore$ need $\delta \int \frac{d\ell}{B} < 0$ for stability

if $\delta \int \frac{d\ell}{B} > 0 \Rightarrow$ critical ∇p for instability exists

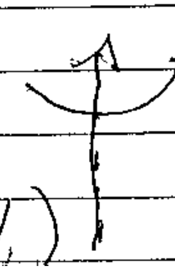
examples

a) wire (i.e. current)

$$d\ell = 2\pi r$$

$$B = 2I/r$$

$$\frac{d\ell}{B} \sim r^2 \quad \text{(unstable!)}$$



i.e. $\delta \int \frac{dl}{B} \sim \delta \int r^2 > 0$

b.) dipole $\quad B \sim 1/r^3$

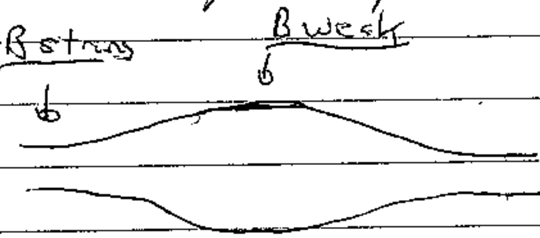
$\qquad dl \sim r$

$dl/B \sim r^4 \Rightarrow$ unstable !

can show $\left| \frac{d\ln P}{d\ln r} \right| < 4\gamma$ is stability criterion

N.B.

→ motivation from early mirror / bumpy torus work

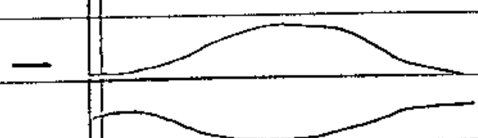


i.e. – simple mirror

$\left.\begin{array}{l} \mu B + \frac{1}{2} m v_\parallel^2 = E \\ \mu = \frac{1}{2} m v_\perp^2 / B \end{array}\right\rangle$ constants

$\Rightarrow$ loss cone

but



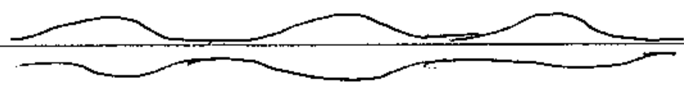
$v_\parallel^2 / R_c \sim$ goes outward

∇P inward $\qquad \Rightarrow$ R–T instability

$\nabla \rho$

50 – fan fields (i.e. Ioffe bars) to reverse gaff

–



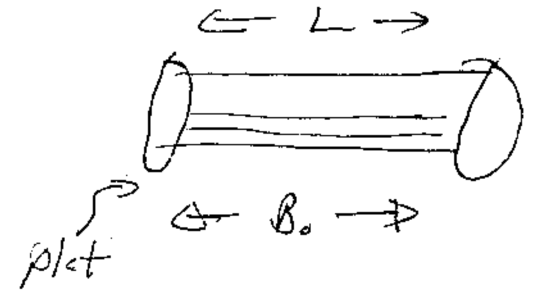
$\int dl/B$ i.e. weight favorable regions more than unfavorable via design.

→ Line Tying and Conducting End Plates

- Till now, have ignored boundary ($k_\parallel = 0$)

⇒ consider plasma between two conducting endplates

i.e.

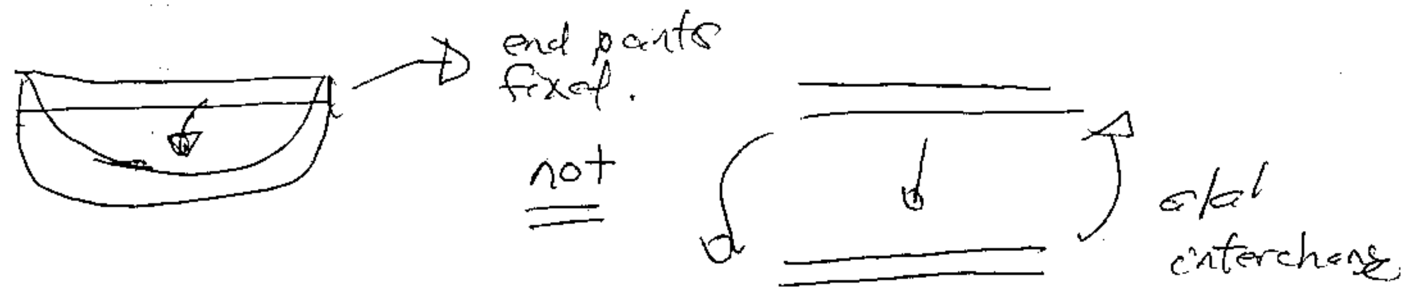


$\underline{E}_t = 0$ on plate

$\Rightarrow \underline{\xi}\Big|_{\text{plate}} = 0$

lines are tied

i.e. displacement has form:



⇒ field lines bent! → Q^2 contribution to δW kinks in!

now, $\underline{J}_0 = 0$ ⇒

for interchanges → expansion free energy

$$\delta W = \int d^3x \left[\frac{Q^2}{8\pi} + \gamma p (\nabla \cdot \underline{\xi})^2 + (\underline{\xi} \cdot \nabla p_0)(\nabla \cdot \underline{\xi}) \right]$$

$$\underline{Q} = \nabla \times \underline{\xi} \times \underline{B}_0$$

$$= \underline{B}_0 \cdot \nabla \underline{\xi} - \underline{\xi} \cdot \nabla \underline{B}_0 - \underline{B}_0 \, \nabla \cdot \underline{\xi}$$

$\nabla \cdot \underline{\xi} \mp 0$ new stabilizing effect → bending! B_0/L finite

$$\delta W = \int d^3x \left[\frac{(\underline{B}_0 \cdot \nabla \underline{\xi} - \underline{B}_0 \nabla \cdot \underline{\xi})^2}{8\pi} + \gamma p (\nabla \cdot \underline{\xi})^2 + (\underline{\xi} \cdot \nabla p_0) \nabla \cdot \underline{\xi} \right]$$

i.e. can't take $\underline{B}_0 \cdot \nabla \underline{\xi} = 0$ any more

→ primary displacement is radial

so $\quad Q \sim B_0 \dfrac{\partial \xi_r}{\partial z}$ $\qquad$ i.e. can make $(\nabla \cdot \underline{\xi}) \underline{B}_0$ smaller ... i.e. $\cancel{\nabla \cdot \underline{\xi} = 0}$.

new

$$\delta W \sim V \left[\frac{B_0^2}{8\pi} \left(\frac{\partial \xi_r}{\partial z} \right)^2 + \underbrace{\gamma p \left(\frac{\delta U}{U} \right)^2 + \delta p \frac{\delta U}{U}}_{\text{old} \to \text{usual interchange terms}} \right]$$

i.e. schematic ....

$$\frac{\partial \xi_r}{\partial z} \sim \frac{\xi_r}{L}$$

L ↓ length

$$\frac{\delta U}{U} = \frac{\nabla U}{U} \xi_r$$

$$\delta p = \nabla p \, \xi_r$$

$\Rightarrow$

$$\delta W \sim V \left\{ \left(\underbrace{\frac{B_0^2}{8\pi L^2}}_{\text{bending}} + \underbrace{\gamma P \left(\frac{\nabla\psi}{\psi}\right)^2}_{\text{compression}} + \underbrace{\left(\nabla P \frac{\nabla\psi}{\psi}\right)}_{\text{interchange}} \right) \varepsilon^2 \right)$$

$\therefore$ $\delta W < 0 \rightarrow$ instability $\Rightarrow$

instability if $-\nabla P \dfrac{\nabla\psi}{\psi} < \gamma P \left(\dfrac{\nabla\psi}{\psi}\right)^2 + \underbrace{\dfrac{B^2}{8\pi L^2}}$

$\Rightarrow$ line tying raises critical pressure gradient

$\Rightarrow$ clearly stabilizing $\Rightarrow$ β limit!

(additional stabilizing effect)

Physics $\rightarrow$ fixing end points forces bending of field lines

$\rightarrow$ lose interchange structure

$\rightarrow$ energy expended coupling to plucking magnetic field lines.

see additional

⇒ key point ⇒ line tying establishes <u>beta limit</u>

<u>i.e.</u>

$$-\frac{\nabla P}{u}\nabla u < \gamma P\left(\frac{\nabla u}{u}\right)^2 + \frac{B_0^2}{8\pi L^2}$$

instability if: $\frac{\nabla u}{u} > 0$ $\qquad \frac{\nabla u}{u} \sim \frac{1}{a}$

<u>now</u> $$\frac{P_{Th}}{a L_p} < \gamma \frac{P_{Th}}{a^2} + \frac{B_0^2}{8\pi L^2}$$

$$\Rightarrow \quad \beta < \frac{a L_p}{L^2}$$ ⇒ simple β-limit criterion

$$a \nabla P_{Th} < \gamma P_{Th} + \left(\frac{B_0^2}{8\pi}\right)\frac{a^2}{L^2}$$